Preferred Schemes for Multiple Orbit Transfer

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A mathematical structure is formulated and used to obtain the preferred sequence of orbits visited for a multiple orbit maneuver between m given coplanar circular orbits. Various constraints are placed on the initial and final orbits visited, and it is assumed that each transfer mode is accomplished via a Hohmann maneuver. The criterion used to extract the class of preferred schemes from the totality of transfer schemes is that the velocity required to perform the maneuver be minimized; phasing considerations are neglected. Although under certain conditions a unique optimal procedure is obtained, in general, the class of possible transfer schemes between m orbits with typical constraints is reduced from $m - 1!$ to 2^{m-2}, preferred schemes. An example of the former arises when the initial orbit visited is the innermost; then all other orbits must be visited, and the final orbit is unconstrained. An example of the latter occurs when it is required to start at the innermost orbit, visit all the other orbits, and then return to the innermost orbit. The mathematical results obtained are developed as a series of theorems, lemmas, and corollaries in order to facilitate application to other problem areas.

Introduction

CONSIDER a transfer vehicle that is maneuvering in an inverse square gravitational field. Suppose that this vehicle is initially in a given circular orbit and must visit each of m other coplanar circular orbits and return to a designated one of these orbits. Further assume that only Hohmann transfer maneuvers will be used, and phasing is not considered. Then what is the preferred order of visit? The material that follows evolved in an attempt to answer this question. This material is divided into two sections. The first of these sections is concerned with the development of a number of mathematical theorems, whereas the second is concerned with the application of these theorems to the solution of the problem just stated. The first section constitutes a self-contained mathematical exposition with potential application beyond that made by the second section. The reader who is interested only in the application of the mathematics to the problem just stated need only read the second section and refer to the first section for the statement of results used and appropriate definitions as needed.

I. Analysis

Consider a finite set of points $A_M = \{a_1, a_2, \ldots, a_m\}$ which is assumed linearly ordered by the relation "less than," indicated by the usual symbol $<$. Let P be a real-valued function defined on $A_M \times A_M$ such that for all $a_i, a_j, a_k, a_r, \in A_M$,

$$P(a_i, a_i) = 0 \tag{1}$$

$$P(a_i, a_j) > 0 \qquad \text{if } j \neq i \tag{2}$$

$$P(a_i, a_j) = P(a_j, a_i) \tag{3}$$

$$P(a_i, a_j) < P(a_i, a_k) + P(a_k, a_j) \qquad \text{if } k \neq j \text{ and } k \neq i \tag{4}$$

$$P(a_i, a_j) < P(a_k, a_r) \qquad \text{if } a_k \leq a_i < a_j < a_r \text{ or } a_k < a_i < a_j \leq a_r \tag{5}$$

The set of equations (1–5) may be thought of as constituting a metric defined over A_M. For brevity, the function P simply will be called the metric. This definition, along with the notation $P(a_i, a_j)$, corresponds heuristically to the cost of traveling from element a_i to element a_j. Some definitions that are basic to the forthcoming discussion are as follows:

Definition: $A \equiv A_M$; $A^k \equiv A \times A \times \ldots \times A$, k times.

Definition: A k chain is defined as an element $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ of A^k for $k \geq 2$ such that $\lambda_j \neq \lambda_{j+1}$, $j = 1,2, \ldots, k-1$. The points λ_j, $1 \leq j \leq k$ of the chain $c = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ will be called elements of c.

Definition: The content $\bar{c}$ of a k chain $c = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is $\bar{c} = \{\lambda_1\} \cup \{\lambda_2\} \cup \ldots \cup \{\lambda_K\}$.

Definition: The points λ_1 and λ_k in the k chain $c = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ will be called the starting and terminal points of c, respectively.

Definition: The metric of a chain $c = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is defined as

$$\sum_{i=1}^{k-1} P(\lambda_i, \lambda_{i+1})$$

and henceforth will be written as $P(c)$.

Definition: A chain $c = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a minimal chain if and only if there does not exist a chain $c_1 = c_1(\lambda_1, \ldots, \lambda_k)$ such that $\bar{c}_1 \supset \bar{c}$ and $P(c_1) < P(c)$.

Definition: $c = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a k chain without repetition if $\lambda_i \neq \lambda_j$ for $j \neq i$, $1 \leq i, j \leq k$.

Definition: If $c = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a k chain and $(b_1, b_2, \ldots, b_k)$ is that permutation of $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ $b_1 < b_2 < \ldots < b_k$, then the k chain $B = (b_1, b_2, \ldots, b_k)$ is called the associated normalized chain.

Some interesting theorems and lemmas related to the metric P and the set A_M now will be stated and proved.

Lemma 1: If a chain c_1 is obtained from the chain $c = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ by deleting some elements, then $P(c_1) < P(c)$.

It clearly suffices to prove this result for the case that c_1 is obtained from c by deleting one element. Let λ_r be the point deleted. If $\lambda_r = \lambda_1$ or λ_k, then the result follows immediately from property 2 of the metric. If $\lambda_r \neq \lambda_1$ or λ_k, then

$$P(c) - P(c_1) = P(\lambda_{r-1}, \lambda_r) + P(\lambda_r, \lambda_{r+1}) - P(\lambda_{r-1}, \lambda_{r+1}) > 0$$

via property 4 of the metric.

Lemma 2: If $c = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a minimal k chain with $\lambda_1 \neq \lambda_k$, then c is a chain without repetition.

The proof of this lemma follows immediately from lemma 1. Assume that there exists $\lambda_j = \lambda_h$ when $j < h$.

If $h < k$, delete λ_h from c. The resulting $(k - 1)$ tuple may fail to be a chain because (and only because) $\lambda_{h-1} =$

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λ_{h+1}. If this is the case, either λ_{h-1} is not the initial point of c (i.e., $h \neq 2$) or λ_{h+1} is not the final point of c (i.e., $h \neq k-1$), for otherwise $\lambda_1 = \lambda_k$. Delete λ_{h-1} if $h \neq 2$. Otherwise, delete λ_{h+1}. The $(k-2)$ tuple resulting from the second deletion now must be a chain, for if it were not, either $\lambda_{h-2} = \lambda_{h-1} = \lambda_{h+1}$ or $\lambda_{h-1} = \lambda_{h+1} = \lambda_{h+2}$. Either of these conditions indicates equality of adjacent elements of c, violating the assumption that c is a chain.

Thus it has been shown that if $\lambda_j = \lambda_h$ with $j < h$ and $h < k$, it is possible to arrive at a chain c_1 with the same end points as c by the deletion of one or two elements of c. Furthermore, in either case, the content of c is unchanged, since the deleted elements were repetitions. Thus one has $\bar{c}_1 \supset \bar{c}$, but by lemma 1, $P(c_1) < P(c)$. This contradicts the minimality of c.

If $h = k$, then $j \neq 1$, for this would contradict the hypothesis $\lambda_1 \neq \lambda_k$. Then the same result follows as before, with c_1 obtained from c by deleting λ_j and possibly one other element.

Theorem 1: Let $c = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a k chain without repetition, and let $B = (b_1, b_2, \ldots, b_k)$ be its associated normalized chain. Then $P(c) \geq P(B)$.

Theorem 1 simply says that the metric of any chain cannot be less than the metric of that chain consisting of the same elements arranged sequentially in increasing order. The proof will be established by employing induction on the number of elements k in c. Note that the theorem obviously is true for $k = 2$. Assume that it is true for $k = m$. When $k = m + 1$, let $c = (\lambda_1, \lambda_2, \ldots, \lambda_m, \lambda_{m+1})$ be an $m + 1$ chain without repetition, and let $B = (b_1, b_2, \ldots, b_m, b_{m+1})$ be its associated normalized chain. Let $\lambda_{m+1} = b_r$ and $\lambda_m = b_s$. Suppose that $1 < r < m + 1$. Then $P(c) - P(B) = P(c_1) - P(B) + P(b_s, b_r)$, where $c_1 = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ is an m chain without repetition. By the induction hypothesis,

$$P(c_1) \geq P(b_1, b_2, \ldots, b_{r-1}, b_{r+1}, \ldots, b_{m+1})$$

This implies

$$P(c) - P(B) \geq P(b_s, b_r) - P(_rb_{-1}, b_r) + P(b_{r-1}, b_{r+1}) - P(b_r, b_{r+1})$$

Noting that

$$s < r \rightarrow P(b_s, b_r) \geq P(b_{r-1}, b_r)$$
$$s > r \rightarrow P(b_s, b_r) \geq P(b_r, b_{r+1})$$

and

$$P(b_{r-1}, b_{r+1}) > P(b_{r-1}, b_r) \quad \text{or} \quad P(b_r, b_{r+1})$$

then

$$P(c) - P(B) > 0$$

If $r = 1$, then by a similar argument $P(c_1) \geq P(b_2, b_3, \ldots, b_{m+1})$, which implies that $P(c) - P(B) \geq P(b_s, b_1) - P(b_1, b_2) \geq 0$, since $s \neq 1$. If $r = m + 1$, then $P(c_1) \geq P(b_1, b_2, \ldots, b_m)$, which implies that $P(c) - P(B) \geq P(b_s, b_{m+1}) - P(b_m, b_{m+1}) \geq 0$, since $s \neq m + 1$.

Theorem 2: If $c = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a minimal k chain such that $\lambda_1 < \lambda_k$, and if $\lambda_1 < \lambda_j < \lambda_k$ for $1 < j < k$, then $\lambda_1 < \lambda_2 < \ldots < \lambda_k$.

Let c satisfy the hypothesis of the theorem. Then from lemma 2, it follows that c is a chain without repetition. The theorem obviously is true for $k = 2,3$. Assume that it is true for $k = m \geq 3$. Let $k = m + 1$ and $c = (\lambda_1, \lambda_2, \ldots, \lambda_m, \lambda_{m+1})$ be a minimal $m + 1$ chain satisfying the hypothesis of the theorem. Since $m + 1 \geq 4$, the set $E = \{\lambda_2, \lambda_3, \ldots, \lambda_m\}$ has at least two elements. Let λ_r be the biggest element of E, and suppose that $r < m$. Then $P(c) = P(\lambda_1, \lambda_2, \ldots, \lambda_m) + P(\lambda_m, \lambda_{m+1})$ and $\lambda_m < \lambda_r$. Let $B = (b_1, b_2, \ldots, b_m, b_{m+1})$ be the associated normalized chain to c. Clearly, $b_1 = \lambda_1$ and $b_{m+1} = \lambda_{m+1}$. By theorem 1, $P(\lambda_1, \lambda_2, \ldots, \lambda_m) \geq P(b_1, b_2, \ldots, b_m)$, which implies that $P(c) \geq P(b_1, b_2, \ldots, b_m) + P(\lambda_m, b_{m+1})$. But $\lambda_m < \lambda_r = b_m$ (since b_m is the largest element in E), which implies that $P(\lambda_m, b_{m+1}) > P(b_m, b_{m+1})$ and hence $P(c) > P(B)$, which contradicts the minimality assumption on c. Therefore $r = m$, and hence λ_m, is the largest element of the m chain $F = (\lambda_1, \ldots, \lambda_m)$. Then $\lambda_1 < \lambda_2 < \ldots < \lambda_m$ by the induction assumption; also $\lambda_m < \lambda_{m+1}$ by hypothesis. The theorem now is established.

Theorem 3: Let $c = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a k chain without repetition, and let $B = (b_1, b_2, \ldots, b_k)$ be its associated normalized chain. Suppose that $P(c) = P(B)$ and $b_1 = \lambda_1$. Then $\lambda_k = b_k$.

The theorem is trivially true for $k = 2$. Let c be a k chain with $k > 2$ satisfying the hypothesis of the theorem and suppose that $\lambda_k \neq b_k$. Let $\lambda_k = b_r$, where $1 < r < k$. The metric of c can be written as $P(c) = P(\lambda_1, \lambda_2, \ldots, \lambda_{k-1}) + P(\lambda_{k-1}, \lambda_k)$. Theorem 1 shows that $P(\lambda_1, \lambda_2, \ldots, \lambda_{k-1}) \geq P(b_1, b_2, \ldots, b_{r-1}, b_{r+1}, \ldots, b_k)$, which implies that $P(c) - P(B) \geq - P(b_{r-1}, b_r) - P(b_r, b_{r+1}) + P(b_{r-1}, b_{r+1}) + P(b_s, b_r) > 0$, where $\lambda_{k-1} = b_s$, by the same argument used in theorem 1. Since this is a contradiction, then $\lambda_k = b_k$.

A corollary that may be verified easily once theorem 3 is known is given now.

Corollary 1: Let $c = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a k chain without repetition such that $\lambda_1 < \lambda_j$, $1 < j \leq k$ and $\lambda_k < \lambda_h$ for some h, $1 < h < k$. Then $P(c) > P(B)$, where B is the associated normalized chain to c.

Theorem 4: Let $c = (\lambda_1, \lambda_2, \ldots, \lambda_1)$ be a minimal k chain ($k \geq 3$) that starts and terminates at its least element λ_1. Let λ_b be the largest element of c. Then $\lambda_1 < \lambda_2 < \ldots < \lambda_b > \lambda_{b+1} > \ldots > \lambda_1$.

This theorem says that c will increase monotonically to its largest element and then decrease monotonically back to λ_1.

Let c and λ_b be as in the hypothesis of the theorem. Then $P(c) = P(\lambda_1, \lambda_2, \ldots, \lambda_b) + P(\lambda_b, \lambda_{b+1}, \ldots, \lambda_1)$. Now $(\lambda_1, \lambda_2, \ldots, \lambda_b)$ and $(\lambda_b, \lambda_{b+1}, \ldots, \lambda_1)$ must be minimal chains. It then follows from theorem 2 that $\lambda_1 < \lambda_2 < \ldots \lambda_b$ and $\lambda_b > \lambda_{b+1} > \ldots > \lambda_1$. In addition, lemma 1 shows that no element other than λ_1 or λ_b of $(\lambda_1, \lambda_2, \ldots, \lambda_b)$ appears in $(\lambda_b, \lambda_{b+1}, \ldots, \lambda_1)$, and vice-versa. Thus c has only one point of repetition, namely, λ_1.

A more general conclusion related to chains with the same starting and terminal points is

Theorem 5: Let $c = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a minimal k chain ($k \geq 3$) that starts and terminates at the same element $\lambda_1 = \lambda_k$. Let $\lambda_b \neq \lambda_1$ be the largest element of c, and let $\lambda_s \neq \lambda_1$ be the smallest element of c. Then

1) $\lambda_1 < \lambda_2 < \ldots < \lambda_b > \ldots > \lambda_s < \ldots < \lambda_k$ if $b < s$
2) $\lambda_1 > \lambda_2 > \ldots > \lambda_s < \ldots < \lambda_b > \ldots > \lambda_k$ if $s < b$

Let c be as in the statement of the theorem, and suppose that $1 < b < s < k$. Then $c = (\lambda_1, \lambda_2, \ldots, \lambda_b, \ldots, \lambda_s, \ldots, \lambda_k)$, where $\lambda_k = \lambda_1$. Let $c_1 = (\lambda_1, \lambda_2, \ldots, \lambda_b)$, $c_2 = (\lambda_b, \ldots, \lambda_s)$, and $c_3 = (\lambda_s, \ldots, \lambda_k)$. It follows from lemma 2 that c_1, c_2, and c_3 are chains without repetition and from theorem 2 that c_2 is strictly monotone decreasing chain. Let $E = (\lambda_s, \lambda_{s+1}, \ldots, \lambda_{k-1}, \lambda_1, \lambda_2, \ldots, \lambda_b)$. Then $P(c) = P(E) + P(c_2)$. Then, in order for c to be minimal, E must be strictly monotone increasing by theorem 2. But this implies relation 1 of the theorem. The case where $b > s$ is proved in a similar manner.

Corollary 2: Let $c = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a minimal k chain ($k \geq 3$) without repetition which starts at its least element and terminates at λ_k, where $\lambda_1 < \lambda_k < \lambda_j$ for all j, $1 < j < k$. Let λ_b be the largest element of c. Then $\lambda_1 < \lambda_2 < \ldots \lambda_b > \ldots > \lambda_k$.

Corollary 3: Let $c = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a minimal k chain ($k \geq 2$) with $\lambda_1 < \lambda_j$, $1 < j \leq k$. Then c may be written as $c = (\lambda_1, \ldots, \lambda_{b_1}, \ldots, \lambda_{r_1}, \ldots, \lambda_{b_2}, \ldots, \lambda_{r_2}, \ldots, \lambda_k)$, where the elements $\lambda_{b_1}, \lambda_{b_2}, \ldots, \lambda_{r_1}, \lambda_{r_2}, \ldots$ can be selected so that 1) $\lambda_{b_1} > \lambda_{b_2} > \ldots \geq \lambda_k$ and $\lambda_{r_1} < \lambda_{r_2} < \ldots \leq \lambda_k$, and 2) the chain $(\lambda_1, \ldots, \lambda_{b_1})$ is monotone increasing, $(\lambda_{b_1}, \ldots, \lambda_{r_1})$ is monotone decreasing, $(\lambda_{r_1}, \ldots, \lambda_{b_2})$ is monotone increasing.

To make this selection for any i, $1 \leq i < k$, define the chain $c_i = (\lambda_{i+1}, \lambda_{i+2}, \ldots, \lambda_k)$. Let λ_{b_1} be the largest element of c_1. Let λ_{r_1} be the smallest element of c_{b_1}. Let λ_{b_2} be the largest element of c_{r_1}; let λ_{r_2} be the smallest element of c_{b_2}, etc. This process continues until some λ_{b_j} or λ_{r_j} is λ_k.

II. Application to Multiple Orbit Transfer Problems

The material presented in the preceding section was motivated by its association with a class of minimal velocity transfer problems. It is our intent here to display this association and to use the theory that has been developed in an effort to answer certain questions of interest within the forementioned class of problems.

The velocity required to execute a so-called Hohmann transfer between two coplanar circular orbits is given by the equation

$$P(r_i, r_j) = \Delta V_{i,j} = \left| \left(\frac{\mu}{r_i}\right)^{1/2} \left[\left(\frac{2r_j/r_i}{1 + r_j/r_i}\right)^{1/2} \times \left(1 - \frac{1}{r_j/r_i}\right) + \frac{1}{(r_j/r_i)^{1/2}} - 1 \right] \right| \quad (6)$$

where r_i and r_j are the initial and final orbit radii, respectively, and μ is the gravitational constant of the earth.[1] The transfer mode that generates Eq. (6) is characterized by an ellipse whose apsides are cotangential to the two circular orbits. Let $A_M = \{r_1, r_2, \ldots, r_m\}$ be a finite set of radii such

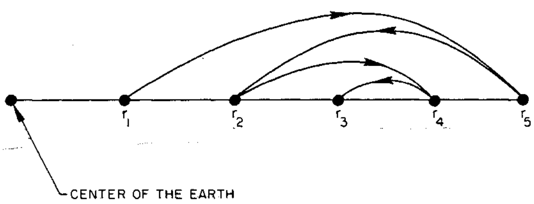

Fig. 1 An allowable transfer scheme (nonadjacent terminal orbits).

that $r_1 < r_2 < \ldots < r_m$ and $r_m/r_1 < 15.6$. It can be shown, with some effort expended, that Eq. (6) satisfies all the conditions noted in Eqs. (1–5).[2, 3] The requirement that $r_m/r_1 < \alpha \sim 15.6$ stems from the fact that $\Delta V_{i,j}$ considered as a function of r_j for fixed r_i is monotone increasing for $r_j < \alpha r_i$ and decreasing for $r_j > \alpha r_i$. In the absence of this requirement, Eq. (5) of Sec. I would not apply to Eq. (6) for all r_j/r_i. This implies that the results of the preceding section can be applied directly to the problem of selecting the appropriate sequence of Hohmann transfers so as to minimize the velocity required to visit a given set of orbits. Some illustrations of the problems which can be handled either completely or in part will be given now.

Consider the case where the transfer vehicle is located at either the innermost or outermost orbit, i.e., r_1 or r_m. It is desired to transfer into each of the orbits without any constraint on the selection of the final orbit. Then by lemma 2, theorem 2, and corollary 1, the optimal selection scheme is to visit each adjacent orbit in turn without any repetition. Note that this indicates that the selection scheme is completely independent of the actual geometrical configuration of the orbits. The velocity required will be given by either

$$\Delta V_T = \Delta V_{1,2} + \Delta V_{2,3} + \ldots + \Delta V_{m-1,m}$$

or

$$\Delta V_T = \Delta V_{m,m-1} + \Delta V_{m-1,m-2} + \ldots \Delta V_{2,1}$$

It also should be noted that the velocity required for both situations is equal. The magnitude of ΔV_T is a function of the individual radii and radii ratios. If for this case it is desired to return to the initial orbit, say r_1, for convenience, then the selection scheme to be used is not determined uniquely by the velocity function. It is known from lemma 2 and theorem 4 that the optimal selection scheme is comprised of a monotone increasing sequence of orbits out to r_m and then a monotone decreasing sequence back to r_1. It also is known that there is no repetition of orbits visited except for r_1. To illustrate, assume the number of orbits to be 4. Then the allowable transfer schemes and associated velocities are

$$\begin{aligned}
\Delta V_{T_1} &= \Delta V_{1,2} + \Delta V_{2,3} + \Delta V_{3,4} + \Delta V_{4,1} \\
\Delta V_{T_2} &= \Delta V_{1,4} + \Delta V_{4,3} + \Delta V_{3,2} + \Delta V_{2,1} \\
\Delta V_{T_3} &= \Delta V_{1,2} + \Delta V_{2,4} + \Delta V_{4,3} + \Delta V_{3,1} \\
\Delta V_{T_4} &= \Delta V_{1,3} + \Delta V_{3,4} + \Delta V_{4,2} + \Delta V_{2,1}
\end{aligned}$$

Although there are four allowable paths, it can be seen that there are only two generally distinct values for ΔV_T, since $\Delta V_{T_1} = \Delta V_{T_2}$ and $\Delta V_{T_3} = \Delta V_{T_4}$. This is a direct consequence of the fact that $\Delta V_{i,j} = \Delta V_{j,i}$. Generally speaking, the number of allowable paths is 2^{m-2}, where m is the number of orbits, whereas the number of distinct values for ΔV_T is 2^{m-3} for $m > 2$. It readily is appreciated that this constitutes a considerable savings in effort over a completely random selection of transfer paths, for which there are $(m - 1)!$ in number even without repetition.

Quite similar to the previous case is the situation that arises when the transfer vehicle is located initially at some arbitrary intermediate orbit and it is desired to visit all the orbits and then return to the starting orbit. From theorem 5, this selection scheme is comprised of either two monotone increasing and one monotone decreasing sequence of orbits or two monotone decreasing and one monotone increase sequence. As before, there is no repetition of orbits other than the starting one. To illustrate, let the number of orbits be four, and let the starting orbit be the second from the innermost; then

$$\begin{aligned}
\Delta V_{T_1} &= \Delta V_{3,4} + \Delta V_{4,1} + \Delta V_{1,2} + \Delta V_{2,3} \\
\Delta V_{T_2} &= \Delta V_{3,2} + \Delta V_{2,1} + \Delta V_{1,4} + \Delta V_{4,3} \\
\Delta V_{T_3} &= \Delta V_{3,4} + \Delta V_{4,2} + \Delta V_{2,1} + \Delta V_{1,3} \\
\Delta V_{T_4} &= \Delta V_{3,1} + \Delta V_{1,2} + \Delta V_{2,4} + \Delta V_{4,3}
\end{aligned}$$

Once again it is seen that two of the ΔV_T's are identical. In general, the number of allowable paths under the theorems will be 2^{m-2}, whereas the number of distinct values for ΔV_T will be one-half of this number. Note that the number of these cases corresponds identically to those given for the case where it is desired to return to the innermost orbit.

There remains a class of multiple orbit transfer problems which has not been analyzed fully. Stated briefly, these problems are concerned with finding the optimum selection of orbits so as to minimize the velocity required to go from an arbitrary orbit r_i to another arbitrary orbit r_j, where at least one of these is not an extreme orbit. What is known of these cases is contained in corollary 2 and corollary 3 of the preceding section.

To illustrate corollary 2, let the takeoff orbit be r_1 and let the final orbit be r_2, i.e., the innermost and the next to the innermost orbits, respectively. Then, according to the corollary, the optimum selection scheme consists of a monotone increasing sequence of orbits out to the outermost orbit, followed by a monotone decreasing sequence back to r_2. No repetition of orbits is allowed. When the number of orbits is four, the velocity required for the two allowable paths is

$$\begin{aligned}
\Delta V_{T_1} &= \Delta V_{1,3} + \Delta V_{3,4} + \Delta V_{4,2} \\
\Delta V_{T_2} &= \Delta V_{1,4} + \Delta V_{4,3} + \Delta V_{3,2}
\end{aligned}$$

The actual path to be chosen depends upon the sign of $\Delta V_{T_1} - \Delta V_{T_2}$ and thus upon the radii and radii ratios involved.

If the initial orbit is r_1 and the final orbit is some arbitrary intermediate orbit other than r_2, then all that is known of the optimum scheme (see corollary 3) is that it is comprised of a set of nested monotone sequences. For instance, the outermost orbit must be visited in the first monotone sequence. If r_2 is not selected in the first sequence, then it must be visited in the second sequence, etc. Figure 1 shows one allowable path between an extreme orbit (r_1) and an intermediate orbit (r_3). Note that each monotone sequence has the maximum possible swing.

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Optimization of Interplanetary Stopover Missions

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A method of interplanetary trip selection is described, using a 10-day orbital stopover mission at Mars as an example. Analysis of numerous possible interplanetary orbits integrated with transportation system design established a set of minimum-mass requirements on Earth parking orbit. The selection of best trips is shown to rely on the relationship between the parameters in transportation system design and mission trajectories. Corresponding trajectory information uniquely defines a best trip once the on-orbit mass is determined. In general, trip selection on the basis of trajectory parameters alone, such as departure velocities, is insufficient for ascertaining the best trip in terms of vehicle mass required.

Scope of Study

An analysis of interplanetary missions involves a multitude of considerations, ranging from transfer trajectories to spacecraft design. Preliminary mission evaluation requires a basis from which an entire system can be defined. This paper presents such a basis.

To develop a method of trip selection which will provide values for the major design parameters, the mass required on Earth orbit was selected as the basic criterion for trip comparison. A best trip is defined as one that minimizes this mass. The primary independent variables are the arrival and departure velocities for both Earth and the destination planet. These are integrated with an assumed transportation system to define the mass on Earth orbit. The absolute magnitudes of the results are secondary and should be used only as representative values.

An orbital stopover at Mars was selected as an example mission to demonstrate the trip selection method. The mission requirements are limited to a three-man crew and a 10-day orbital stopover at Mars for the August 10, 1971 opposition. Various other assumptions are made as to the type of planetary entry, propulsion systems, structural factors, and life support systems.

The method is presented in sufficiently complete form so as to be adaptable to other applications, such as different reentry systems, different oppositions and conjunctions, advanced system concepts, and more ambitious missions.

Technical Approach

The interorbital trajectory data used in this study are taken from Ref. 1. A chart for the Mars opposition of August 10, 1971 is given in Figs. 1a and 1b. This chart displays four sets of hyperbolic excess speed contours normalized with respect to Earth's mean orbital speed: one for departure from Earth, one for arrival at Mars, one for departure from Mars, and one for arrival at Earth. The outbound departure and arrival speed contours are superimposed on Fig. 1a, whereas the inbound departure and arrival contours are superimposed on Fig. 1b. All departure contours are shown as solid lines, whereas all arrival contours are shown as broken lines.

The contour charts are read in the following manner. Suppose one wishes to depart from Earth on 244 0880 and arrive at Mars on 244 1130. The required hyperbolic excess speed of departure is then 0.3 EMOS (Earth's mean orbital speed), and the arrival speed at Mars is 0.2 EMOS (Fig. 1a, outbound). If the stay time at the destination planet is one month, one then leaves at about 244 1160 (Fig. 1b, inbound). If one elects to return home on 244 1280, the Mars departure speed is thus 0.3 EMOS, and the arrival speed at Earth is 0.2 EMOS. Of course, the inbound and outbound leg durations and total trip time is the difference between the appropriate Julian dates. The basic assumptions for the example computation are listed below for reference:

1) August 10, 1971 opposition, 10-day orbital stopover at Mars with no manned landing.

2) O_2/H_2 Earth escape propulsion, estimated pre-1970 state of technology, storable propulsion system for Mars escape.

3) Drag brake of Apollo-type command module at Earth (three men), drag-brake capture at Mars.

4) Arrival velocity limitations: 50,000 fps at Earth, 40,000 fps at Mars. Earth departure from 200-km orbit, Mars departure from 500-km orbit.

The Earth drag-brake system consists of an Apollo-type command capsule, a large drag brake, and a propulsion system to provide vehicle lift (Fig. 2). The capsule is decelerated from its approach speed down to 36,000 fps, the designed entry speed for Apollo. The Mars capture system also consists of a drag brake and a propulsion system to augment

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